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DLL-003-016407 Seat No. ________

M. Sc. (Mathematics) (Sem. IV) (CBCS) Examination

May / June – 2015

Maths CMT - 4001 : Linear Algebra

(New Course)

Faculty Code : 003

Subject Code : 016407

Time : Hours] [Total Marks : 70

Instructions : (1) Answer all the questions.
(2) Each question carries 14 marks.
(3) Unless otherwise specified, vector spaces consdiered here are finite - dimensional.

1. Answer any **Seven** $7 \times 2 = 14$

(a) Define an algebra over a field F and illustrate it with an example.

(b) When is an element $\lambda \in F$ said to be a characteristic root of $T \in A_F(V)$? Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be given by $T(1,0) = (1,1)$ and $T(0,1) = (1,-1)$. Determine the characteristic roots of T.

(c) Let $n \geq 1$ and $A \in F_n$. Define $tr(A)$. Verify that the map $tr : F_n \to F$ is linear.

(d) State Cramer's rule.

(e) Let $(V, <, >)$ be an inner product space over $\mathbf{C}$. If a linear transformation $T : V \to V$ is unitary, then prove that $T^*T = Id_V$.

(f) Let $T \in A_F(V)$. When is a subspace W of V said to be invariant under T? Verify that $T^2(V)$ is a subspace of V and is invariant under T.

(g) Let $T \in A_F(V)$ be nilpotent. Define the concept of the invariants of T.

(h) Let V be a vector space over $\mathbf{R}$. Let $f : V \times V \to \mathbf{R}$ be bilinear. When is f said to be skew-symmetric? If f is skew-symmetric, then show that $f(v,v) = 0$ for any $v \in V$.

(i) Let V be a vector space over a field F and let $\mathcal{B} = \{v_1, \dots, v_n\}$ be a basis of V over F. Let $T \in A_F(V)$. Define $[T]_\mathcal{B}$, the matrix of T in $\mathcal{B}$. For any $T, S \in A_F(V)$, verify that $[T+S]_\mathcal{B} = [T]_\mathcal{B} + [S]_\mathcal{B}$.

(j) Let $n \geq 1$ and $A \in F_n$. Define the secular equation of A. Let $A \in F_3$ be any diagonal matrix. Determine the secular equation of A.

2. Answer any **Two** $2 \times 7 = 14$

(a) Let V be a finite-dimensional vector space over a field F. Let $T \in A_F(V)$. Prove the following:

(i) If T is invertible, then T^{-1} is a polynomial expression in T over F.

(*ii*) If T is singular, then there exists a nonzero $S \in A_F(V)$ such that $ST = TS =$ zero map from V into V.
(*b*) If A is an algebra, with unit element, over a field F, then prove that A is isomorphic to a subalgebra of $A_F(V)$ for some vector space V over F.
(*c*) If $T \in A_F(V)$ has all its characteristic roots in F, then prove that there is a basis of V in which the matrix of T is triangular.

3. (*a*) Let $\lambda \in F$ be a characteristic root of $T \in A_F(V)$. Prove that for any polynomial $q(X) \in F[X]$, $q(\lambda)$ is a characteristic root of $q(T)$. 5
(*b*) If V is finite-dimensional over F and if $T \in A_F(V)$ is onto, then prove that T is regular. 5
(*c*) If $T \in A_F(V)$, then prove that $tr(T)$ is the sum of the characteristic roots of T (using each characteristic root as often as its multiplicity). 4

OR

3. (*a*) Let $A \in F_n$. If two rows of A are equal, then prove that $det A = 0$. 5
(*b*) Let $(V, <,>)$ be a finite-dimensional inner product space over $\mathbf{C}$. Let $T \in A_{\mathbf{C}}(V)$. Then given $v \in V$, prove that there exists an element $w \in V$, depending on v and T such that $< T(u), v >=< u, w >$ for all $u \in V$. 5
(*c*) Let V be as in (*b*). Let $T, S \in A_{\mathbf{C}}(V)$. Prove that $(ST)^* = T^*S^*$. 4.

4. Answer any **Two** $2 \times 7 = 14$
(*a*) Let $T \in A_F(V)$ be nilpotent and let n_1 be its index of nilpotence. Let $v \in V$ be such that $T^{n_1-1}(v) \neq 0$. Let V_1 be the subspace of V spanned by the vectors $v, T(v), \ldots, T^{n_1-1}(v)$. Then prove the following:
(*i*) $dim_F(V_1) = n_1$.
(*ii*) If $u \in V_1$ is such that $T^{n_1-k}(u) = 0$ for some k with $0 < k \leq n_1$, then $u = T^k(u_0)$ for some $u_0 \in V_1$.
(*b*) Let $(V, <,>)$ be a finite-dimensional inner product space over $\mathbf{C}$. Let $T \in A_{\mathbf{C}}(V)$. Prove that T is unitary if and only if T maps an orthonormal basis of V into an orthonormal basis of V.
(*c*) Let $f : V \times V \to F$ be a bilinear form on an n-dimensional vector space V over F. If $\mathcal{B}$, $\mathcal{B}'$ are any two basis of V over F, then prove that there exists an invertible matrix $C \in F_n$ such that $[f]_{\mathcal{B}'} = C[f]_{\mathcal{B}}C'$.

5. Answer any **Two** $2 \times 7 = 14$
(*a*) Prove that any $A \in F_n$ satisfies its secular equation.
(*b*) Let $(V, <,>)$ be as in 4(*b*). If $N \in A_{\mathbf{C}}$ is normal, then prove that there exists an orthonormal basis of V consisting of characteristic vectors of N, in which the matrix of N is diagonal.
(*c*) Let V be a finite-dimensional vector space over a field F and let $T \in A_F(V)$. Prove that T is invertible if and only if the constant term of the minimal polynomial for T over F is nonzero.
(*d*) (*i*) Let $A, B \in F_n$. Prove $tr(AB) = tr(BA)$.
(*ii*) Let F be a field of characteristic 0. If $T, S \in A_F(V)$ are such that $ST - TS$ commutes with S, then prove that $ST - TS$ is nilpotent.
